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ON GENERALIZATIONS OF SOME COMBINATORIAL IDENTITIES

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Dedicated to Prof. M.A. Pathan on his 75^{th} ***birth anniversary***

Abstract: In this paper, using split $(n+t)-$color partitions, R-weighted lattice paths and modified lattice paths, we interpret two q-series which leads to new 3-way combinatorial identities. These generalize some of the results due to Agarwal and Sachdeva.

Keywords: Split $(n+t)-$color partitions, lattice paths, combinatorial interpretation, combinatorial identities.

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1. Introduction

A partition of an integer n is a non-increasing sequence of positive integers whose sum is n. An elementary device for studying partitions is the graphical representation. Many combinatorial objects such as lattice paths, Ferrers graphs etc are useful to represent partitions graphically. A lattice path P is a sequence $P = (a_0, a_1, a_2, \cdots, a_k)$ of points a_i in $\mathbb{Z}^d$, $0 \le i \le k$. The point a_0 is the starting point and the point a_k is the terminating point of the path P. The vectors $\overrightarrow{a_0a_1}, \overrightarrow{a_1a_2}, \cdots, \overrightarrow{a_{k-1}a_k}$ are called the steps of the path P. Throughout this paper, we consider the paths in the plane integer lattice $\mathbb{Z}^2$.

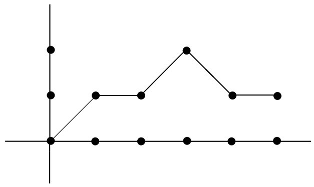

Figure 1:

For example, the above graph represents the lattice path P=((0,0), (1,1), (2,1), (3,2), (4,1), (5,1)).

Long time ago, counting lattice paths with various restrictions considered. For a detailed history see [8]. Lattice paths have applications in many fields of mathematics, physics and computer science. In particular, lattice paths are used as a simple combinatorial object to interpret many $q-$series identities combinatorially. In 1989, Agarwal and Bressoud [1] introduced a new class of weighted lattice paths. They interpreted certain basic hypergeometric series with multiple indices of summation as generating functions for weighted lattice paths. In the same paper they established a bijection between the appropriate class of lattice paths of weight n and a set of colored partitions of n. Now, we recall the definitions of $(n+t)$-color partitions (also called the partitions with "$n+t$ copies of n"), weighted difference [2] and split $(n+t)$-color partitions [5].

Definition 1.1 *A partition with "$n+t$ copies of n", $t \geq 0$, is a partition in which a part of size n, $n \geq 0$, can occur in (n+t) different colors denoted by subscripts $n_1, n_2, \cdots, n_{n+t}$.*

Note that zeros are permitted if and only if t is greater than or equal to one and in no partition zeros are permitted to repeat.

Definition 1.2 *The weighted difference of two elements m_i and n_j, $m \geq n$, is defined by $m-n-i-j$ and is denoted by $((m_i - n_j))$.*

Definition 1.3 *Let m_i be a part in an $(n+t)$-color partition of a non-negative integer ν. We split the color 'i'into two parts-the green part and the red part and denote them by 'g' and 'r', respectively, such that $1 \leq g \leq i$, $0 \leq r \leq i-1$ and $g+r=i$. An $(n+t)$-color partition in which each part is split in this manner is called a split $(n+t)$-color partition.*

Using the ordinary partitions, color partitions, Frobenius partitions and weighted lattice paths, many researchers interpreted several basic series identities combinatorially (see for example [4, 7]). Recently Sachdeva and Agarwal [9] introduced a new combinatorial object which they called modified lattice path and they interpreted following two eight-order mock theta functions of Gordon and McIntosh [6] combinatorially using modified lattice paths.

$$V_0(q) = 1 + 2\sum_{n=1}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q;q^2)_n}, \tag{1.1}$$

$$V_1(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}(-q;q^2)_{n-1}}{(q;q^2)_n}, \tag{1.2}$$

where $(a;q)_n = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1})$, $|q|<1$.
Earlier, Agarwal and Sood [5] interpreted (1.1) and (1.2) combinatorially using split n-color partitions. Very recently, Agarwal and Sachdeva [3] interpreted the following two basic series identities combinatorially in three different ways-using ordinary partitions, split (n+t)-color partitions and the modified lattice paths which leads to new 3-way combinatorial identities:

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{2n^2}}{(q;q^2)_n (q^4;q^4)_n} = \prod_{n=1}^{\infty} \frac{(1+q^{6n-3})^2}{(1-q^{6n-2})(1-q^{6n-4})}$$

and

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{2n(n+1)}}{(q;q^2)_{n+1} (q^4;q^4)_n} = \prod_{n=1}^{\infty} \frac{(1+q^{6n-1})(1+q^{6n-5})}{(1-q^{6n-2})(1-q^{6n-4})}.$$

In this paper, we consider the following two $q-$series:

$$\sum_{n=0}^{\infty} \frac{q^{kn^2}(-q^l;q^{2l})_n}{(q^p;q^{2p})_n (q^t;q^t)_n} \tag{1.3}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{kn^2+rn}(-q^l;q^{2l})_n}{(q^p;q^{2p})_{n+1} (q^t;q^t)_n}, \tag{1.4}$$

where k, l, p, r and t be any positive integers with $t \geq 2$. We shall prove that (1.3) and (1.4) are the generating functions of three different combinatorial objects- a split (n+t)-color partition function, R-weighted lattice path function and modified lattice path function. This leads to two new 3-way combinatorial identities which generalize the results due to Agarwal and Sachdeva [3].

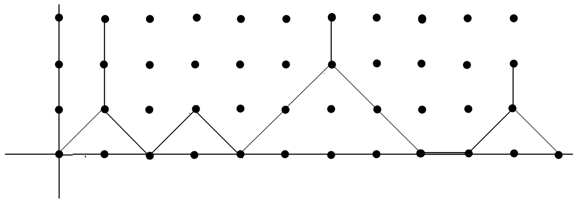

Figure 2:

2. R-weighted lattice paths

In this section using weighted lattice paths [1], we introduce a new class of lattice paths called R-weighted lattice paths. For definitions of peak, valley, mountain and plain, we refer to [1].

We also need the following terminologies to describe R-weighted lattice paths.

Vertical line: It is a line from (a, b) to $(a, b + k)$ allowed only on the peaks, $k \geq 0$.

Peak with head: It is a peak and there is a vertical line of finite length starting from peak and lying above the peak. Note that vertical lines allowed only on peaks except peaks on y-axis.

Peak without head is just a peak and its weight is its x-coordinate and weight of a peak with head is the sum of its x-coordinate and height of its head.

Definition 2.1 *R-weighted Lattice path: It is a weighted lattice path wherein vertical lines of finite length (length may be zero) lying above the peaks and one is not allowed to walk on head of the peaks.*

Weight of R-weighted lattice path is the sum of weights of its peaks with head or without head.

In the Figure 2, there are three peaks of height one, one peak of height two and there are two valleys of height zero. Length of the first head is two and so weight of first peak with head is three. Similarly, weights of second, third and forth peak with head is three, seven and eleven respectively. The weight of this R-weighted lattice path is $3 + 3 + 7 + 11 = 24$.

3. Main results

In section, we state the main results of this paper.

Theorem 3.1 *Let k, l, p and t be any positive integers such that $t \geq 2$. Let $E_m(k, l, p, t)$ denote the number of R-weighted Lattice paths of weight m which start from $(0, 0)$ such that each peak of height $\equiv k \pmod{p}$, have no valley above height 0, length of the plain if any in the path is $\equiv 0 \pmod{t}$ and height of i^{th} head is* *either* $0,\ l, \cdots, (2i-3)l$ *or* $(2i-1)l$. *Then*

$$\sum_{m=0}^{\infty} E_m(k,l,p,t)q^m = \sum_{n=0}^{\infty} \frac{q^{kn^2}(-q^l;q^{2l})_n}{(q^p;q^{2p})_n(q^t;q^t)_n}.$$

Theorem 3.2 *Let* k,l,r, p *and* t *be any positive integers such that* $t \geq 2$. *Let* $F_m(k,l,p,r,t)$ *denote the number of R-weighted Lattice paths of weight* m *which start from* $(0,r)$ *such that the height of first peak is* $\geq r$ *and if any other peak is of height* $\equiv k \pmod p$, *have no valley above height* 0, *the lengths of the plains if any in the path are* $\equiv 0 \pmod t$, *first peak must be with head* 0 *height and height of* $(i+1)^{th}$ *head is either* $0,\ l, \cdots, (2i-3)l$ *or* $(2i-1)l$, *where* $1 \leq i \leq n$. *Then*

$$\sum_{m=0}^{\infty} F_m(k,l,p,r,t)q^m = \sum_{n=0}^{\infty} \frac{q^{kn^2+rn}(-q^l;q^{2l})_n}{(q^p;q^{2p})_{n+1}(q^t;q^t)_n}.$$

Theorem 3.3 *Let* k,l *and* p *be any positive integers and for any even* $t \geq 2$, *let* $A_m(k,l,p,t)$ *denote the number of modified lattice paths of wight* m *such that*
1) they start from $(0,0)$;
2) they have no valley above hight 0;
3) the length of plains if any are $\equiv 0 \pmod t$;
4) the height of beam is either 0 *or* l;
5) the height of each pillar is $\equiv k \pmod p$.

Let $B_m(k,l,p,t)$ *denote the split n-color partitions of* m *such that*
1) the parts and their subscripts have the same parity;
2) if n_i *is the smallest or the only part in the partition, then* $n \equiv i \pmod t$;
3) the weighted difference of any two consecutive parts is non-negative and $\equiv 0 (mod\ t)$;
4) the red part is either 0 *or* l;
5) the green part is $\equiv k \pmod p$.
Then for all non-negative integer m, *we have*

$$A_m(k,l,p,t) = B_m(k,l,p,t) = E_m(k,l,p,t).$$

Theorem 3.4 *Let* k,l,r *and* p *be any positive integers and for any even* $t \geq 2$, *let* $C_m(k,l,p,r,t)$ *denote the number of modified lattice paths of wight* m *such that*
1) they start from $(0,r)$;
2) they have no valley above hight 0;
3) the length of plains if any are $\equiv 0 \pmod t$;

4) the height of beam is either 0 *or* l*;*
5) the height of each pillar is $\equiv k \pmod p$*;*
6) the first peak is supported by a pillar only.

Let $D_m(k,l,p,r,t)$ *denote the split* n*-color partitions of* m *such that*
1) the parts and their subscripts have the same parity;
2) if n_i *is the smallest or the only part in the partition, then* $n \equiv r+i \pmod t$*;*
3) the weighted difference of any two consecutive parts is non-negative and $\equiv 0(mod\ t)$*;*
4) for some i, i_{i+r} *is a part;*
5) the red part is either 0 *or* l*;*
6) the green part is $\equiv k \pmod p$*;*
7) the red part of the smallest part is 0*.*
Then for all non-negative integer m*, we have*

$$C_m(k,l,p,r,t) = D_m(k,l,p,r,t) = F_m(k,l,p,r,t).$$

Theorem 3.3 and Theorem 3.4 are the generalizations of the results proved by Agarwal and Sachdeva [5]. In fact, setting $k=2, l=p=1, t=4$ in Theorem 3.3, we obtain the Theorem 2.1 of [3] and setting $k=2, l=p=1, t=4$ and $r=2$ in Theorem 3.4, we obtain the Theorem 2.2 of [3].

4. Proof of Theorem 3.1–3.4

Proof of Theorem 3.1. In $\dfrac{q^{kn^2}(-q^l;q^{2l})_n}{(q^p;q^{2p})_n(q^t;q^t)_n}$, the factor q^{kn^2} generates the lattice path with n peaks and height of each peak is k starting from $(0,0)$ and terminate at $(2nk,0)$. The factor $\dfrac{1}{(q^t;q^t)_n}$ generates n non-negative multiples of t say, $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ \cdots \lambda_n \geq 0$, which are encoded by inserting λ_n horizontal steps in front of the first mountain and $\lambda_i - \lambda_{i+1}$ horizontal steps in front of the $(n-i+1)^{th}$ mountain, $1 \leq i \leq n-1$. The factor $\dfrac{1}{(q^p;q^{2p})_n}$ generates n non-negative multiples of $(2i-1)p$ say, $b_1 \times p, b_2 \times 3p, b_3 \times 5p, \cdots, b_n \times (2n-1)p$. These can be encoded by raising the height of i^{th} peak by $b_{n-i+1}p, 1 \leq i \leq n$. The factor $(-q^l;q^{2l})_n$ generates non-negative multiples of distinct $(2i-1)l$ say, $\beta_1 \times l, \beta_2 \times 3l, \cdots, \beta_n \times (2n-1)l$, where $\beta_i \in \{0,1\}$, $1 \leq i \leq n$. This is encoded by inserting vertical line of length $2l(\beta_n + \beta_{n-1} + \cdots + \beta_{n-i+2}) + l\beta_{n-i+1}$ on the i^{th} peak towards above, where $1 \leq i \leq n$. Every lattice paths enumerated by $E_m(k,l,p,t)$ is uniquely generated in this manner.

Proof of Theorem 3.2. The proof of Theorem 3.2 is similar to the proof of

Theorem 3.1. We see that in this case there are two extra factors namely q^{rn} and $(1-q^{(2n+1)p})^{-1}$. The factor q^{rn} puts r south east steps: $(0,r)$ to $(1,r-1),\cdots,(r-1,1)$ to $(r,0)$. Thus, there are $n+1$ peaks starting from $(0,r)$ and the extra factor $(1-q^{(2n+1)p})^{-1}$ generates a non-negative multiple of $(2n+1)p$, say $b_{n+1}\times p(2n+1)$. This can be inserted by raising the height of the first peak by pb_{n+1}. This completes the proof.

Proof of Theorem 3.3. In $\dfrac{q^{kn^2}(-q^l;q^{2l})_n}{(q^p;q^{2p})_n(q^t;q^t)_n}$, the factor q^{kn^2} generates the lattice path with n peaks and each peak is supported by a pillar of height k starting from $(0,0)$ and terminating at $(2nk,0)$. The factor $\dfrac{1}{(q^t;q^t)_n}$ generates n non-negative multiples of t say, $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n\geq 0$. These can be encoded by inserting λ_n horizontal steps in front of the first mountain and $\lambda_i-\lambda_{i+1}$ horizontal steps in front of the $(n-i+1)^{th}$ mountain, $1\leq i\leq n-1$. The factor $\dfrac{1}{(q^p;q^{2p})_n}$ generates n non-negative multiples of $(2i-1)p$ say, $b_1\times p$, $b_2\times 3p,\cdots,b_n\times(2n-1)p$. These can encoded by raising the height of i^{th} pillar by $b_{n-i+1}p$, for $1\leq i\leq n$. The factor $(-q^l;q^{2l})_n$ generates non-negative multiples of distinct $(2i-1)l$ say, $\beta_1\times l$, $\beta_2\times 3l,\cdots,\beta_n\times(2n-1)l$, where $\beta_i=0$ or 1 for $1\leq 1\leq n$. This is encoded by inserting a beam of height $\beta_{n-i+1}l$ on the i^{th} pillar for $1\leq i\leq n$.Every lattice paths enumerated by $A_m(k,l,p,t)$ is uniquely generated in this manner. Now, we shall prove 1–1 correspondence between the modified lattice paths enumerated by $A_m(k,l,p,t)$ and the split-n color partitions enumerated by $B_m(k,l,p,t)$. We do this by encoding each path as the sequence of the weights of the peaks with each weight subscripted by the height of the respective peak considered as the height of the supporting pillar which corresponds to the green color plus the height of the supporting beam which corresponds to the red color. Thus if in the final graph we denote the i^{th} and $(i+1)^{th}$ peak by R_x and S_y $(S\geq R)$, respectively, then

$$\begin{aligned}
R&=(2i-1)k+\lambda_{n-i+1}+2p(b_n+b_{n-1}+\cdots+b_{n-i+2})+pb_{n-i+1}\\
&\quad+2l(\beta_n+\beta_{n-1}+\cdots+\beta_{n-i+2})+l\beta_{n-i+1},\\
x&=k+pb_{n-i+1}+l\beta_{n-i+1},\\
S&=(2i+1)k+\lambda_{n-i}+2p(b_n+b_{n-1}+\cdots+b_{n-i+1})+pb_{n-i}\\
&\quad+2l(\beta_n+\beta_{n-1}+\cdots+\beta_{n-i+1})+l\beta_{n-i},\\
y&=k+pb_{n-i}+l\beta_{n-i}.
\end{aligned}$$

The weighted difference of these two parts is $((S_y-R_x))=S-R-x-y=\lambda_{n-i}-\lambda_{n-i+1}$ which is non-negative and $\equiv 0 \pmod t$. Note that if R_x denote the first peak in the modified lattice path then it will correspond to the smallest part in the corresponding split $n-$ color partition or to the singleton part if split $n-$color partition has only one part and in both cases

$$R - x = \lambda_n \equiv 0 \pmod t.$$

Further, if we look at the split $n-$ color part R_x, we find that the parity of both R and x is determined by $k + pb_{n-i+1} + l\beta_{n-i+1}$. If $k + pb_{n-i+1} + l\beta_{n-i+1}$ is even (resp., odd), then R and x are even (resp., odd). This proves the parts and their subscripts have the same parity. Since the height of any beam is either 0 or l, the red part in the corresponding split n-color partition is either 0 or l.Also, since there is no pillar with height $< k$, the green part in the corresponding split $n-$color partition will be at least k. Since the lengths of the plains are given in terms of λ_i, $1 \leq i \leq n$, which are non-negative and are multiples of t, the lengths of plain, if any, are $\equiv 0 \pmod t$.

To see the reverse implication, we consider the two parts of a partition enumerated by $A_m(k, l, p, t)$, say, C_u and D_v. (Note that here there is no need to consider the split subscripts). Let $Q_1 \equiv (C, u)$ and $Q_2 \equiv (D, v)$ be the corresponding peaks in the associated lattice path.

The length of the plain between the two peaks is $D - C - u - v$ which is the weighted difference between the two parts C_u and D_v and is therefore non-negative and $\equiv 0 \pmod t$. Next, we can prove by contradiction that there cannot be any valley above height 0.Suppose there is a valley V of height r $(r > 0)$ between the peaks Q_1 and Q_2. In this case, there is a descent of $u - r$ from Q_1 to V and an ascent of $v - r$ from V to Q_2. This implies that $D = C + (u - r) + (v - r)$ or $D - C - u - v = -2r$. But since the weighted difference is non-negative, we have $r = 0$. This completes the bijection.

Finally, Note that weight of i^{th} peak in modified lattice path is equal to weight of i^{th} peak with head in R-weighted lattice path for $1 \leq i \leq n$. So, there is a 1–1 corresponds between modified lattice paths enumerated by $A_m(k, l, p, t)$ and R-weighted lattice paths enumerated by $E_m(k, l, p, t)$. This completes the proof.

Proof of Theorem 3.4. The proof of Theorem 3.4 is similar to the proof of Theorem 3.3. We see that in this case there are two extra factors namely q^{rn} and $(1 - q^{(2n+1)p})^{-1}$. The factor q^{rn} puts r south east steps: $(0, r)$ to $(1, r - 1), \cdots, (r - 1, 1)$ to $(r, 0)$. Thus, there are $m + 1$ peaks starting from $(0, r)$ and the extra factor $(1 - q^{(2n+1)p})^{-1}$ introduces a non-negative multiple of $(2n + 1)p$, say $b_{m+1} \times p(2n + 1)$. This is encoded by having the first peak to grow to height $pb_{m+1} + r$. Clearly, $(pb_{m+1})_{pb_{m+1}+r}$ which is of the form i_{i+r} will be the colored part corresponding to the first peak. This completes the proof.

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